Now equating the coefficients of $\theta^{2n+1}$, we have, on expanding $\theta\sin\theta-\cos\theta$,

$$\frac{n\pi^{2n}}{(2n+1)!}=\Sigma_2\cdot\frac{\pi^{2n-2}}{(2n-1)!}-\Sigma_4\cdot\frac{\pi^{2n-4}}{(2n-3)!}+\ldots-(-1)^n\Sigma_{2n},$$

where $\Sigma_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$ to $\infty$.

Hence for $n=1$, $\Sigma_2=\frac{\pi^2}{3!}$, which is $(a)$;

$$n=2,\ \Sigma_2\cdot\frac{\pi^2}{3!}-\Sigma_4=\frac{2\pi^4}{5!}\text{ and }\Sigma_4=\frac{\pi^4}{90},\text{ which is }(c);$$

$$n=3,\ \Sigma_2\cdot\frac{\pi^4}{5!}-\Sigma_4\cdot\frac{\pi^2}{3!}+\Sigma_6\cdot\frac{1}{1!}=\frac{3\pi^6}{7!}.$$

$$\therefore\Sigma_6=\frac{-\pi^6}{3!\,5!}+\frac{\pi^6}{90.3!}+\frac{3\pi^6}{7!}=\frac{\pi^6}{945}.$$

$(r)$ If 1, $a$, $b$, $c$, ..., be all the prime numbers, the numbers not divisible by the square of any prime (unity excepted) are $a$, $b$, $c$, ..., $ab$, $ac$, ..., $abc$, ... Then, the sum of the squares of the reciprocals of all numbers not divisible by the square of any prime is

$$1+a^{-2}+b^{-2}+c^{-2}+\ldots+a^{-2}b^{-2}+\ldots+a^{-2}b^{-2}c^{-2}+\ldots=\Pi(1+a^{-2})=15/\pi^2\text{ by }(e).$$

$(s)$ *Lemma.* To find the limit of $\frac{\sin\theta}{1-(\theta^2/\pi^2)}$ when $\theta=\pi$. Put $\theta=\pi+h$.

$$\frac{\sin\theta}{1-(\theta^2/\pi^2)}=\frac{\sinh}{(2\pi h+h^2)/\pi^2}=\frac{\sinh}{h}\cdot\frac{1}{2/\pi+h/\pi^2}=\tfrac{1}{2}\pi\text{ for }h=0.$$

Now $\frac{\sin\theta}{\theta}=\prod_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\pi^2}\right)$. On dividing both sides by all factors of the type $1-\frac{\theta^2}{\lambda^2\pi^2}$ where $\lambda$ is a prime, we have

$$\frac{\sin\theta}{\theta[1-(\theta^2/\pi^2)][1-(\theta^2/2^2\pi^2)]\ldots}=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\ldots$$

Now if $\theta=\pi$ this becomes

$$\tfrac{1}{2}\left(1-\frac{1}{2^2}\right)^{-1}\left(1-\frac{1}{3^2}\right)^{-1}\ldots=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\ldots$$

The left hand $=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\right)\ldots$

$$=\tfrac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)=\frac{\pi^2}{12} \text{ by } (a).$$

Taking logarithms of both sides

$$2\log\pi-\log 12=\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots+\tfrac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\ldots\right)+\tfrac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\ldots\right)+\ldots$$

$$=\Sigma_2+\tfrac{1}{2}\Sigma_4+\tfrac{1}{3}\Sigma_6+\ldots$$

where $S_r$ is the sum of the reciprocals of the $r$th powers of all numbers that are not prime.

$(t)$ Equate coefficients of $\theta^8$ in $(A)$

$$-\frac{1}{4\pi^8}\Sigma_8=\frac{1}{9!}-\tfrac{1}{2}\left(\frac{1}{5!\,5!}+\frac{2}{3!\,7!}\right)+\tfrac{1}{3}\cdot\frac{3}{3!\,3!\,5!}-\tfrac{1}{4}\frac{1}{(3!)^4}=-\tfrac{1}{4}\cdot\frac{1}{9450}.$$

$$\therefore \Sigma_8=\frac{\pi^8}{9450}.$$

$(u)$ $$\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{1}{n^4}\left\{\sum_{n=1}^{n=\infty}\frac{1}{n^4}\right\}\left(1-2.\frac{1}{2^4}\right)=\frac{\pi^4}{90}\cdot\frac{7}{8}=\frac{7\pi^4}{720}.$$

$(v)$ $$\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{1}{n^8}\left(\sum_{n=1}^{n=\infty}\frac{1}{n^8}\right)\left(1-2.\frac{1}{2^8}\right)=\frac{7\pi^4}{720}\times\frac{127}{128}.$$

$(w)$ $$\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^8}=\left(\sum_{n=1}^{n=\infty}\frac{1}{n^8}\right)\left(1-\frac{1}{2^8}\right)=\frac{\pi^8}{9450}\times\frac{255}{256}=\frac{17\pi^8}{256\times 630}.$$

$(x)$ $$\sum_{n=1}^{n=\infty}\frac{4}{n^2(n+3)^2}=\tfrac{4}{9}\sum_{n=1}^{n=\infty}\left(\frac{1}{n^2}+\frac{1}{(n+3)^2}\right)$$

$$-\tfrac{8}{9}\sum_{n=1}^{n=\infty}\left(\frac{1}{n(n+1)}-\frac{2}{n(n+1)(n+2)}+\frac{2}{n(n+1)(n+2)(n+3)}\right)$$

$$=\tfrac{4}{9}\left(\frac{\pi^2}{6}+\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}\right)-\tfrac{8}{9}\left(1-\frac{2}{4}+\frac{2}{18}\right)=\frac{4\pi^2-31}{27}.$$

$(y)$ $$\frac{2}{1^3+2^3}+\frac{3}{3^3+4^3+5^3}+\frac{4}{6^3+7^3+8^3+9^3}+\ldots$$

$$=\sum_{n=1}^{n=\infty}\frac{8}{n^2(n+1)^2(n+2)^2}$$

$$= \sum_{n=1}^{n=\infty} \left( \frac{2}{n^2} + \frac{8}{(n+1)^2} + \frac{2}{(n+2)^2} - \frac{12}{n(n+1)} + \frac{12}{n(n+1)(n+2)} \right)$$

$$= \frac{12\pi^2}{6} - 8 - 2\left(\frac{1}{1^2} + \frac{1}{2^2}\right) - 12 + 3 = 2\pi^2 - 19\tfrac{1}{2}.$$

(*z*) $$\frac{1}{1^2.2^2} - \frac{3}{4^2.5^2} + \frac{5}{7^2.8^2} + \ldots = \tfrac{1}{3}\left[1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \ldots\right]$$

$$= \tfrac{1}{3}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \ldots\right]$$

$$- \frac{1}{3^3}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right] = \left(\frac{1}{3} - \frac{1}{3^3}\right)\frac{\pi^2}{12} \text{ by } (i) = \frac{2\pi^2}{81}.$$

(*a*) From the identity

$$\frac{1}{(4n-3)^4\,(4n-1)^4} = \frac{1}{2^4}\left[\frac{1}{(4n-3)^4} + \frac{1}{(4n-1)^4}\right] - \frac{1}{2^3}\left[\frac{1}{(4n-3)^3} - \frac{1}{(4n-1)^3}\right]$$

$$+ \frac{5}{2^5}\left[\frac{1}{(4n-3)^2} + \frac{1}{(4n-1)^2}\right] - \frac{5}{2^5}\left[\frac{1}{4n-3} - \frac{1}{4n-1}\right]$$

we have

$$\frac{1}{1^4.3^4} + \frac{1}{5^4.7^4} + \frac{1}{9^4.11^4} + \ldots = \sum_{n=1}^{n=\infty} \frac{1}{(n-3)^4(n-1)^4}$$

$$= \frac{\pi^4}{96\times 2^4} - \frac{1}{2^3}\cdot\frac{\pi^3}{32} + \frac{5}{2^5}\cdot\frac{\pi^2}{8} - \frac{5}{2^5}\cdot\frac{\pi}{4} = \frac{1}{1536}(\pi^4 - 6\pi^3 + 30\pi^2 - 60\pi).$$

To find $\frac{1}{1^4.3^4} + \frac{1}{3^4.5^4} + \frac{1}{5^4.7^4} + \ldots = \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^4\,(2n+1)^4}$ we use the identity

$$\frac{1}{(2n-1)^4\,(2n+1)^4} = \frac{1}{2^4}(u^4+v^4) - \frac{1}{2^3}(u^3-v^3) + \frac{5}{2^5}(u^2-v^2)$$

$$- \frac{5}{2^5}(u-v) \text{ where } u = \frac{1}{2n-1},\ v = \frac{1}{2n+1}.$$

Hence the sum required is found to be $\frac{1}{768}(\pi^4 + 30\pi^2 - 384)$.

(*ζ*) Taking logarithms in (*A*) and equating coefficients of $\theta^4$, and also of $\theta^8$ we have $\sum_{n=1}^{n=\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{n=\infty} \frac{1}{n^8}$ as in (*c*) and (*t*).

Now $$2\sum_{m=1}^{m=\infty} \frac{1}{m^4} \sum_{n=1}^{n=\infty} \frac{1}{n^4} = \left(\sum_{n=1}^{n=\infty} \frac{1}{n^4}\right)^2 - \sum_{n=1}^{n=\infty} \frac{1}{n^8}$$

$$= \frac{\pi^8}{8100} - \frac{\pi^8}{9450} \text{ by } (c) \text{ and } (t) = \frac{2\pi^8}{113400}.$$

$$\therefore \sum_{m=1}^{m=\infty} \frac{1}{m^4} \sum_{n=1}^{n=\infty} \frac{1}{n^4} = \frac{384\pi^8}{9!\,5!}$$

and this is the sum of the products of the fourth powers of the reciprocals of every pair of positive integers.

($\gamma$) Consider the identity $\dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{2.n} - \dfrac{1}{n+1} + \dfrac{1}{2(n+2)}$.

We have $\dfrac{1}{n^2(n+1)^2(n+2)^2} = \dfrac{1}{4}\cdot\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \dfrac{1}{4(n+2)^2}$

$$-\frac{1}{n}\cdot\frac{1}{(n+1)} + \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{1}{n+2} - \frac{1}{(n+1)(n+2)} = \ldots - \frac{3}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

$$\therefore \sum_{n=1}^{n=\infty} \frac{1}{n^2(n+1)^2(n+2)^2} = \tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{n^2} + \tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{(n+1)^2} + \tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{(n+2)^2}$$

$$-\tfrac{3}{4}(1-\tfrac{1}{3}+\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{4}-\tfrac{1}{6}-\ldots)$$

$$= \frac{\pi^2}{24} + \frac{\pi^2}{6} - 1 + \frac{1}{4}\left(\frac{\pi^2}{6} - 1 - \frac{1}{4}\right) - \frac{3}{4}\cdot\frac{3}{2}$$

$$= \frac{\pi^2}{4} - \frac{39}{16}.$$

($\delta$) Since, when $a, b, c, \ldots$ are all the primes save unity

$\prod_2^\infty (1-a^{-8})^{-1} = \dfrac{\pi^8}{9450}$, and $\prod_2^\infty (1-a^{-4})^{-1} = \dfrac{\pi^4}{90}$ by ($c$) and ($t$)

$$\prod_2^\infty (1+a^{-4})^{-1} = \frac{\pi^4}{105}, \text{ i. e., } \frac{2^4}{2^4+1}\cdot\frac{3^4}{3^4+1}\cdot\frac{5^4}{5^4+1}\cdots = \frac{\pi^4}{105}.$$

($\varepsilon$) Since $\sum_{n=1}^{n=\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, $\sum_{u=1}^{n=\infty}\dfrac{1}{(3n)^2} = \dfrac{\pi^2}{54}$.

Subtracting, we have $\dfrac{4\pi^2}{27} = 1 + \dfrac{1}{2^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots$ where the denominators are the squares of all numbers not divisible by 3,

Euler's *Introductio in Analysis Infinitorum* (1748) Vol. I, Caput X. *De usu Factorum inventorum in definiendis summis Serierum infinitarum*, will be found a happy hunting-ground for those who want more of these series. The Latin is easily comprehended, the reader soon getting used to *Dignitates, Potestates, Exponentes*, and the like, while the methods are worth studying carefully by all interested in the historical development of mathematics.

# A NOTE CONCERNING MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.*

By DR. GILBERT AMES BLISS.

In the discussions of maxima and minima of functions of a single variable, it is usually assumed that the function $f(x)$ under consideration has $n$ continuous derivatives in the neighborhood of a point $a$ at which

$$f'(a)=f''(a)=\ldots=f^{(n-1)}(a)=0 \text{ while } f^{(n)}(a)\neq 0.\dagger$$

The proofs usually given of the corresponding theorems for functions of several variables demand the existence and continuity of order $n+1$, that is of an order one higher than that of the first derivatives which do not vanish at the point under investigation.‡ In the following paragraphs a method is given for functions of several variables in which only the derivatives of the first $n$ orders are used, as in the case of a single independent variable.

Suppose that all the derivatives of the function $f(x_1, x_2, \ldots, x_k)$ up to and including those of the $n$th order exist and are continuous in a vicinity

$$V(a): \qquad a_i-\delta \leqq x_i \leqq a_i+\delta \ (i=1, 2, \ldots, k),$$

of the point $(a_1, a_2, \ldots, a_k)$. If the derivatives of order less than $n$ all vanish at $(a_1, a_2, \ldots, a_k)$ then by Taylor's formula

$$(1)\quad f(a_1+h_1, a_2+h_2, \ldots, a_k+h_k)-f(a_1, a_2, \ldots, a_k)$$
$$=\sum \frac{h_1^{a_1}}{a_1!}\cdot\frac{h_2^{a_2}}{a_2!}\cdots\frac{h_k^{a_k}}{a_k!}\cdot\frac{\partial^n f}{\partial x_1^{a_1}\,\partial x_2^{a_2}\ldots\partial x_k^{a_k}},$$

where the summation is for all sets of positive integers $a_1, a_2, \ldots, a_k$ whose sum is $n$, and the arguments of the derivatives of $f$ are $a_i+\theta h_i$ $(i=1, 2, \ldots, k)$, $0<\theta<1$. If the right member is divided by $r^n$, where

$$r=\sqrt{[h_1^2+h_2^2+\ldots+h_k^2]},$$

it becomes a homogeneous quadratic form $H_n$ of degree $n$ in the $k$ ratios $\eta_i=\frac{h_i}{r}$ which satisfy the equation

$$(2)\qquad \eta_1^2+\eta_2^2+\ldots+\eta_k^2=1.$$

Equation (1) may therefore be written

*Read at the Chicago Meeting of the American Mathematical Society, December 30, 1906.

†See for example, Pierpont, *The Theory of Functions of Real Variables*, p. 318.

‡Pierpont, *loc. cit.*, p. 322, *ff.*

$$f(a_1+h_1,\ a_2+h_2,\ \ldots,\ a_k+h_k)-f(a_1,\ a_2,\ \ldots,\ a_k)=r^n H_n(\eta,\ a+\theta h),$$

where $\eta,\ a+\theta h$ stand for the rows of letters $(\eta_1,\ \eta_2,\ \ldots,\ \eta_k)$ and $(a_1+\theta h_1,\ a_2+\theta h_2,\ \ldots,\ a_k+\theta h_k)$, respectively.

When the values $a+\theta h$ in $H_n$ are replaced by the values $a$ and the $\eta$'s are allowed to take any values satisfying equation (2), the quadratic form $H_n(\eta,\ a)$ belongs to one of the following classes:

*a*) It is indefinite, *i. e.*, there exist values of $\eta$ for which it has opposite signs; or

*b*) It is semi-definite, *i. e.*, it does not take on opposite signs but there exist some values of $\eta$ for which it vanishes; or

*c*) It is definite, *i. e.*, for all values of $\eta$ it has the same sign.

*If the form $H(\eta,\ a)$ is indefinite, the function $f$ can not have an extreme at the point $(a_1,\ a_2,\ \ldots,\ a_k)$.*

The proofs of this statement and also of the theorems below depend upon the fact that the function $H(\eta,\ a+h)$ is a continuous function for all values of $\eta$ and for all values of $h_1,\ h_2,\ \ldots,\ h_k$ defining points in the vicinity $V(a)$. Suppose that $\eta'$ is a set of values for which $H(\eta',\ a)$ is negative. On account of its continuity $H(\eta,\ a+h)$ will also be negative for all points $a+h$ in a certain neighborhood $V'(a)$. The equations $h'_i=r\eta'_i$ may now be taken as defining the values $h'$, and $r>0$ can be restricted so that all the points $a+h'$ lie in $V'(a)$. Then it is evident that for the points $a+h'$ the expression (1) is negative. In a similar way, by taking values of $\eta''$ at which $H(\eta'',\ a)>0$, points $a+h''$ can be found in any neighborhood of $a$ such that the difference in (1) is positive.

*If the form $H_n(\eta,\ a)$ is definite, then the function $f$ has a minimum at the point $(a_1,\ a_2,\ \ldots,\ a_k)$ when $H_n$ is positive, and a maximum when it is negative.*

If $H_n(\eta,\ a)$ is definite, a neighborhood $V'(a)$ can be found such that $H_n(\eta,\ a+h)$ is also different from zero for all sets of values $\eta$ satisfying (2), and for all points $a+h$ in $V'(a)$. This follows because $H_n(\eta,\ a+h)$ is continuous, and positive over the closed set of points $(\eta,\ h)$ of $2k$ dimensions which satisfy the equation (2) and $h=0$. Hence in $V'(a)$ the difference in (1) is positive or negative according as $H_n(\eta,\ a)$ is positive or negative.

The function $H_n(\eta,\ a)$ cannot vanish for all sets of values $\eta$ unless the derivatives of $f$ of order $n$ are all zero at the point $a$. It follows that when $n$ is odd, $H_n(\eta,\ a)$ is either identically zero or indefinite, for if $\eta'$ represents values for which it is different from zero, $H(-\eta',\ a)=-H(\eta',\ a)$. It is possible, then, to state the preceding results in the following form:

*A function $f(x_1,\ x_2,\ \ldots,\ x_k)$ is given which with its first $n$ derivatives is continuous in a certain neighborhood $V(a)$ of the point $(a_1,\ a_2,\ \ldots,\ a_k)$. The derivatives of the first $n-1$ orders are zero at the point $(a_1,\ a_2,\ \ldots,\ a_k)$, while sum at least of order $n$ are different from zero. Then if $n$ is odd, or if $n$ is even and the form*

$$H_n(\eta, a)=\sum \frac{\eta_1^{a_1}}{a_1!}\cdot\frac{\eta_2^{a_2}}{a_2!}\cdots\frac{\eta_k^{a_k}}{a_k!}\,\frac{\partial^n f}{\partial x_1^{a_1}\partial x_2^{a_2}\ldots\partial x_n^{a_n}}$$

*is indefinite at the point* $(a_1, a_2, \ldots, a_k)$, *the function* $f$ *has neither a maximum nor a minimum at the point* $(a_1, a_2, \ldots, a_k)$. *If* $n$ *is even and* $H_n(\eta, a)$ *is positive definite the function* $f$ *has a minimum, while if* $H_n(\eta, a)$ *is negative definite* $f$ *has a maximum.*

The case, $\eta$ even and $H_n(\eta, a)$ semi-definite, as usual requires a more elaborate investigation.

PRINCETON, *December, 1906.*

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## DIVIDING BY ZERO.

By DR. G. A. MILLER.

Since the seventh century of our era the Hindus have considered division by zero,* and in the twelfth century the noted Hindu mathematician and astronomer Bhaskara gave the rule that the quotient obtained by dividing a number by zero is not changed by adding even a large number to it or by substracting a large number from it. The same thought is expressed in modern times by "the quotient obtained by dividing a number which is not zero by zero is infinite." This rule is commonly understood to mean, explained by Krishna, a commentator of Bhaskara, that if we divide a given number which is not zero by a number which is small the quotient may be made to exceed any finite number if the divisor is made sufficiently small.

According to this interpretation $a/0$ need not mean the same thing as $\frac{a}{b-b}$. The matter may be made perfectly clear by adopting the notation used by Professor Pierpont in his recent work on the Theory of Functions. If $a$ and $b$ are two distinct numbers then there is an infinity of numbers which do not differ any more from $a$ than $b$ does. The totality of these numbers are said to form the domain of $a$, whose norm is $a\sim b=\rho$. This totality or aggregate of numbers is denoted by $D_\rho(a)$. It is sometimes desirable to exclude $a$ from its domain. In this case the domain is said to be deleted and it is denoted by $D_\rho{}^*(a)$. In particular, $D_\rho{}^*(0)$ means all the numbers which differ from zero by not more than $\rho$, with the exception of zero itself. By making $\rho$ sufficiently small we obtain the aggregate of numbers which are commonly considered when we think of the meaning of $a/0$,

*Cf. Algebra with arithmetic and mensuration from the Sanscrit of Brahmegupta and Bhascara translated by H. T. Colebrooke. London, 1817, p. 137.

so that this symbol is merely an abbreviation of the more rational symbol $a/D_\rho{}^*(0)$.

While all are aware that division by zero, which was allowed by some of the most eminent older mathematicians, Euler for example, has caused a great deal of confusion and hence is commonly ruled out in modern mathematics, there are still some such practices in elementary mathematics which are apt to lead to confusion. It is the aim of this note to point to two instances of this kind and to suggest a method of avoiding the difficulties.

The first of these may be illustrated by the equations

$$\frac{1}{x}+\frac{1}{y}=1,\ x+y=xy.$$

The second of these is obtained by clearing the first of fractions. No one will question the geometric interpretation of the second as it is a common form of the equation of a hyperbola passing through the origin. This hyperbola contains only one point whose co-ordinates do not satisfy the former of the given equations, viz., the origin. As there is no other point whose co-ordinates satisfy this equation we have the interesting result that the loci of $\frac{1}{x}+\frac{1}{y}=1$ and $x+y=xy$ differ only with respect to a single point.

If the loci of these two equations were drawn accurately the microscope of highest power would not make it possible to observe any difference since such an instrument could not exhibit the lacuna caused by the missing point. Hence some might at first be inclined to believe that for practical purposes it would make no difference whether we should call these equations equivalent or not. That this is far from the truth follows directly from the solution of such systems of simultaneous equations as

$$\begin{array}{ll}\frac{1}{x}+\frac{1}{y}=1, & x+y=xy,\\ x=y, & x=y.\end{array}$$

It is evident that the former of these systems is satisfied by only one pair of values for $x$ and $y$ while the latter is satisfied by two pairs, viz., (0, 0), $(\frac{1}{2}, \frac{1}{2})$. Moreover, the method suggested in the elementary algebras for the solution of such a system as

$$\frac{a}{x}+\frac{b}{y}=1,\ \frac{c}{x}+\frac{d}{y}=1,$$

would not give us a complete solution of the system if the distinction indicated in the preceding paragraph were not observed.

The other point to which we wish to call attention in this connection is that the trigonometric equation $\tan 90^\circ=\infty$ is open to serious objection, for

$\tan A = \frac{\sin A}{\cos A}$, and hence $\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$; but, as we cannot divide by 0, we must say that $\tan 90^\circ$ has no value unless we assign to it some arbitrary value indepently of the fact that $\tan A = \frac{\sin A}{\cos A}$. Suppose that $A$ is equal to some number in $D_\rho{}^*(\frac{1}{2}\pi)$, where $\rho$ represents a small number. As $\rho$ is decreased the smallest possible value of $\tan A$ is increased and we can always select $\rho$ in such a way that $\tan A$ must be larger than any arbitrary number. This fact may be expressed by the equation

$$\tan D_\rho{}^*(\tfrac{1}{2}\pi) = \infty \text{ when } \rho \doteq 0.$$

It seems very unfortunate that we should tell the student of elementary algebra that it is not permitted to divide by 0 and then in trigonometry tell him that $\tan \frac{\pi}{2} = \frac{1}{0} = \infty$. It appears to be much better to say that $\tan A$ has a meaning for all values of $A$ except when $A = k\frac{1}{2}\pi$. For these special values, it has no meaning but it becomes indefinitely large when $A$ is very nearly equal to a number of the form $k\frac{1}{2}\pi$. It is evident that similar remarks apply to $\cot A$, $\sec A$, and $\csc A$. Trigonometry appears to be in special need of being freed from the hazy terminology which appealed to the Hindu mind of a thonsand years ago. The main conclusions which have been reached in the above considerations are:

While we may divide by every number in $D_\rho{}^*(0)$ we are not allowed to divide by every number in $D_\rho(0)$. The two domains differ only with respect to one number and if they could be represented geometrically the microscope of highest power in existence would not reveal the lacuna in $D_\rho{}^*(0)$ caused by the omission of 0. If we multiply by a factor to clear of fractions and then make this factor equal to zero, it is equivalent to dividing by zero and hence this operation cannot be allowed. It is better to say that $a/0$ has no meaning in elementary mathematics than to assign it an arbitrary value since the student cannot appreciate the need of such arbitrary values until he is more mature. There are many to whom the ordinary treatment of the trigonometric functions of $k\frac{1}{2}\pi$ appear objectionable and it is probable that the introduction of the concepts of *domain* and *deleted domain* would tend towards greater clearness in this part of the trigonometry.

# ON A CERTAIN QUARTIC CURVE WHICH MAY DEGENERATE INTO AN ELLIPSE.*

By R. D. CARMICHAEL.

The object of this paper is the discussion of a class of quartic curves whose equation may be put in the form

$$\nu\sqrt{[x^2+y^2]} \pm \mu\sqrt{[x^2+(a-y)^2]} = \lambda. \tag{1}$$

When the equation is cleared of radicals the double sign $\pm$ can be made to disappear entirely. In the following discussion it will be shown that this curve consists of two ovals. As a limiting case, when $\mu=\nu$ one of the ovals vanishes and the other becomes an ellipse.

## §1. Construction of the Locus by Continuous Motion.

Take two points $A$ and $B$ at a distance $a$ apart, and let $A$ be the origin of rectangular axes, and let the $y$-axis pass through $B$. Take the point $P$ such that

$$\mu.PB+\nu.PA=\lambda. \tag{2}$$

Then $P$ is a point on the locus; for evidently it satisfies the equation when the double sign is taken *plus*. Hence the branch on which $P$ is located may be defined as the locus of a point which moves so that the sum of $\mu$ times its distance from one point and $\nu$ times its distance from another is constant. We may therefore devise the following method for constructing this branch when $\mu$ and $\nu$ are commensurable.

Divide equation (1) by such common factor of $\mu$ and $\nu$ as will make the respective quotients $m$ and $n$ relatively prime integers, and let the quotient of $\lambda$ divided by this factor be $l$. Then the equation becomes

$$n\sqrt{[x^2+y^2]} \pm m\sqrt{[x^2+(a-y)^2]} = l. \tag{3}$$

For the point $P$ we now have

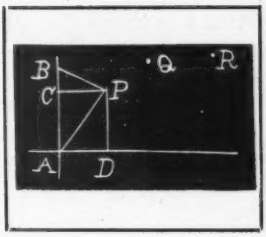
$$m.PB+n.PA=l. \tag{4}$$

One or both of the numbers $m$, $n$ are odd. Consider $m$ odd and $n$ even.

Attach a string at $B$ and pass it around a pencil point at $P$, wrapping until $m$ plies of the cord pass from $B$ to $P$. Then pass the free end around $A$, wrapping until $n$ plies pass from $A$ to $P$. Then attach the cord to the

*Presented to the Chicago Section of the American Mathematical Society, March 30, 1907.

pencil at $P$ so that every part of it is drawn tight. Now if the pencil so moves that the chord is kept tightly stretched while it slides around $A$, $B$, and $P$, $P$ will describe one branch of the curve. (The modifications necessary for $n$ odd or $m$ even are evident). We shall next give a method for constructing the other branch by continuous motion. For this case we shall employ




$$n\sqrt{[x^2+y^2]}-m\sqrt{[x^2+(a-y)^2]}=l. \tag{5}$$

If $Q$ in the figure is so taken that

$$n.AQ-m.BQ=l, \tag{6}$$

$Q$ is evidently a point on this second branch.

Now, pass a cord around the point at $B$ and the pencil at $Q$ so that $m$ plies pass from $B$ to $Q$. If $m$ is odd one end of the string is to be attached at $B$; while if $m$ is even it is to be fastened to the pencil point at $Q$. The free end is to pass out in some direction as $QR$. In like manner join $Q$ and $A$ with another chord having its free end in the direction $QR$. Of this cord $n$ plies are to pass from $A$ to $Q$. Let these two chords be tightly stretched and tied together at some convenient point. If $Q$ moves so as to keep these conditions always fulfilled, it is evident that the pencil will mark on the paper the second branch of this curve.

From the manner of construction it is evident that each of these branches is closed. Such a branch we shall as usual call an oval. In the next section we shall show that these two are the only branches of the locus.

## §2. General Nature of the Locus.

As the two branches of the locus have the respective equations

$$\nu\sqrt{[x^2+y^2]}+\mu\sqrt{[x^2+(a-y)^2]}=\lambda,$$
$$\nu\sqrt{[x^2+y^2]}-\mu\sqrt{[x^2+(a-y)^2]}=\lambda,$$

it is evident they have no point in common except when

$$\mu\sqrt{[x^2+(a-y)^2]}=0.$$

Now $\mu$ is not zero; for then the locus would reduce to a circle. Hence

$$x^2+(a-y)^2=0.$$

Since each term of the first member is positive the equation can be satisfied only for $x^2=0$ and $(a-y)^2=0$. Hence, $x=0$, $y=a$. That is, the two

branches of the curve can have no point in common except $(0, a)$. Moreover, this point is on the locus only when $\nu a = \lambda$, as may be seen by substituting in equation (1). When this condition is satisfied the two branches are tangent at the point $(0, a)$; for if they should cut each other here they would necessarily have another point in common. Moreover one branch is always entirely within the other.

Now if there is another branch to the locus besides the two already discussed some line passing through the inner of these two ovals cuts this third branch. The line then intersects the *quartic* curve in *five* points; but this is impossible. Hence the entire locus consists of the two ovals above constructed.

PRESBYTERIAN COLLEGE, ANNISTON, ALABAMA.

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## DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**276. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.**

If $x_1, x_2, x_3, \ldots, x_n$ be unequal, and $f(x)$ be a rational integral function of degree $\geq n-2$, then shall

$$\sum_{r=1}^{r=n-1} \frac{f(x_r)}{(x_r-x_1)(x_r-x_2)\ \ldots\ (x_r-x_n)} = 0.$$

No solution of this problem has been received. It is not clear what is intended in the problem. By taking $f(x) = x^2 - x + 1$ and $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$, the problem is not true according to our interpretation of it, since infinite addends would be introduced.

In Burnside and Panton's *Theory of Equations*, p. 319, 3rd edition, example 4, slightly changed, reads: If the degree of $\phi(x)$ does not exceed $n-2$, prove that $\sum_{r=1}^{r=n} \frac{\phi(x_r)}{f'(x_r)} = 0$, $\ldots$, $x_r (r = 1, 2, \ldots, n)$ being roots of $f(x) = 0$.

Additional information on this problem is desired. ED. F.

**277. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

If $\alpha, \beta, \gamma, \delta$ are the roots of the quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$, calculate the value of the product of the twelve expressions of the form $(4\alpha - 2\beta - \gamma - \delta)$ in terms of $H$, $I$, $J$, the well known functions of the differences of the roots.

**Solution by the PROPOSER.**

To avoid fractions write the equation in the form $ax^4+4b'x^3+6c'x^2+4d'x+e=0$. See Burnside and Panton's *Theory of Equations*, 4th edition, Vol. I, pp. 121-131.

$H=ac'-b'^2$ $I=ae-4b'd'+3c'^2$, $J=ac'e+2b'c'd'-ad'^2-eb'^2-c'^3$.

$$\left.\begin{aligned} M_1^2&=a^2\theta_1+b'^2-ac'=a^2\theta_1-H\\ M_2^2&=a^2\theta_2+b'^2-ac'=a^2\theta_2-H\\ M_3^2&=a^2\theta_3+b'^2-ac'=a^2\theta_3-H \end{aligned}\right\}\ldots(1),$$

where $\theta_1$, $\theta_2$, $\theta_3$ are the roots of the reducing cubic $4a^3\theta^3-Ia\theta+J=0$.

$$\left.\begin{aligned} a\alpha+b'&=-M_1+M_2+M_3\\ a\beta+b'&=M_1-M_2+M_3\\ a\gamma+b'&=M_1+M_2-M_3\\ a\delta+b'&=-M_1-M_2-M_3 \end{aligned}\right\}\ldots(2).$$

Now $\theta_1+\theta_2+\theta_3=0$, $\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3=-I/4a^2$, $\theta_1\theta_2\theta_3=-J/4a^3$.

The twelve expressions of the form $(4\alpha-2\beta-\gamma-\delta)$ can be easily expressed in terms of $M_1$, $M_2$, $M_3$ from (2), thus

$$\begin{aligned}
\tfrac{1}{2}a(4\alpha-2\beta-\gamma-\delta)&=-[3(M_1-M_2)-2M_3]\\
\tfrac{1}{2}a(4\alpha-\beta-2\gamma-\delta)&=-[3(M_1-M_3)-2M_2]\\
\tfrac{1}{2}a(4\alpha-\beta-\gamma-2\delta)&=[3(M_2+M_3)-2M_1]\\
\tfrac{1}{2}a(4\beta-2\alpha-\gamma-\delta)&=[3(M_1-M_2)+2M_3]\\
\tfrac{1}{2}a(4\beta-\alpha-2\gamma-\delta)&=-[3(M_2-M_3)-2M_1]\\
\tfrac{1}{2}a(4\beta-\alpha-\gamma-2\delta)&=3(M_1+M_3)-2M_2\\
\tfrac{1}{2}a(4\gamma-2\alpha-\beta-\delta)&=3(M_1-M_3)+2M_2\\
\tfrac{1}{2}a(4\gamma-\alpha-2\beta-\delta)&=3(M_2-M_3)+2M_1\\
\tfrac{1}{2}a(4\gamma-\alpha-\beta-2\delta)&=3(M_1+M_2)-2M_3\\
\tfrac{1}{2}a(4\delta-2\alpha-\beta-\gamma)&=-[3(M_2+M_3)+2M_1]\\
\tfrac{1}{2}a(4\delta-\alpha-2\beta-\gamma)&=-[3(M_1+M_3)+2M_2]\\
\tfrac{1}{2}a(4\delta-\alpha-\beta-2\gamma)&=-[3(M_1+M_2)+2M_3]
\end{aligned}$$

These twelve can be reduced to three as follows: $\frac{1}{16}a^4(4\alpha-2\beta-\gamma-\delta)(4\beta-2\alpha-\gamma-\delta)(4\gamma-\alpha-\beta-2\delta)(4\delta-\alpha-\beta-2\gamma)$

$$\begin{aligned}
&=[9(M_1-M_2)^2-4M_3^2][9(M_1+M_2)^2-4M_3^2]\\
&=[9M_1^2+9M_2^2-4M_3^2-18M_1M_2][9M_1^2+9M_2^2-4M_3^2+18M_1M_2]\\
&=[9a^2\theta_1-9H+9a^2\theta_2-9H-4a^2\theta_3+4H-18M_1M_2]\ [9a^2\theta_1-9H+9a^2\theta_2\\
&\qquad -9H-4a^2\theta_3+4H+18M_1M_2]\\
&=[9a^2(\theta_1+\theta_2+\theta_3)-13a^2\theta_3)-14H-18M_1M_2][9a^2(\theta_1+\theta_2+\theta_3)\\
&\qquad -13a^2\theta_3-14H+18M_1M_2]\\
&=(13a^2\theta_3+14H+18M_1M_2)(13a^2\theta_3+14H-18M_1M_2)\\
&=(13a^2\theta_3+14H)^2-324M_1^2M_2^2\\
&=(13a^2\theta_3+14H)^2-324(a^2\theta_1-H)(a^2\theta_2-H)
\end{aligned}$$

$$=169a^4\theta_3^2+364a^2\theta_3H+196H^2-324a^4\theta_1\theta_2+324a^2\theta_1H+324a^2\theta_2H$$
$$-324H^2$$
$$=169a^4\theta_3^2+40a^2\theta_3+324a^2H(\theta_1+\theta_2+\theta_3)-324a^4\theta_1\theta_2-128H^2$$
$$=169a^4\theta_3^2+40a^2\theta_3+81aJ/\theta_3-128H^2$$
$$=\frac{1}{\theta_3}(169a^4\theta_3^3+40a^2\theta_3^2H+81aJ-128H^2\theta_3)=\frac{1}{\theta_3}(A+81aJ).$$

Similarly,

$$\tfrac{1}{16}a^4(4\alpha-\beta-2\gamma-\delta)(4\beta-\alpha-\gamma-2\delta)(4\gamma-2\alpha-\beta-\delta)(4\delta-\alpha-2\beta-\gamma)$$
$$=\frac{1}{\theta_2}(169a^4\theta_2^3+40a^2\theta_2^2H+81aJ-128H^2\theta_2)=\frac{1}{\theta_2}(B+81aJ).$$
$$\tfrac{1}{16}a^4(4\alpha-\beta-\gamma-2\delta)(4\beta-\alpha-2\gamma-\delta)(4\gamma-\alpha-2\beta-\delta)(4\delta-2\alpha-\beta-\gamma)$$
$$=\frac{1}{\theta_1}(169a^4\theta_1^3+40a^2\theta_1^2H+81aJ-128H^2\theta_1)=\frac{1}{\theta_1}(C+81aJ).$$

Let $\Pi$=the product of the twelve factors.

$$\therefore\frac{a^{12}\Pi}{4096}=\frac{1}{\theta_1\theta_2\theta_3}[531441a^3J^3+6561a^2J^2(A+B+C)+81aJ(AB+AC$$
$$+BC)+ABC]$$
$$=-\frac{4a^3}{J}[531441a^3J^3+6561a^2J^2(A+B+C)+81aJ(AB+AC+BC)+ABC].$$
$$A+B+C=169a^4(\theta_1^3+\theta_2^3+\theta_3^3)+40a^2H(\theta_1^2+\theta_2^2+\theta_3^2)-128H^2(\theta_1+\theta_2+\theta_3).$$
$$AB+AC+BC=28561a^8(\theta_1^3\theta_2^3+\theta_1^3\theta_3^3+\theta_2^3\theta_3^3)+1600a^4H^2(\theta_1^2\theta_2^2+\theta_1^2\theta_3^2$$
$$+\theta_2^2\theta_3^2)+16384H^4(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)+6760a^6H(\theta_1^2\theta_2^3+\theta_1^3\theta_2^2+\theta_1^2\theta_3^3$$
$$+\theta_1^3\theta_3^2+\theta_2^2\theta_3^3+\theta_2^3\theta_3^2)-21632a^4H^2(\theta_1^3\theta_2+\theta_1\theta_2^3+\theta_1^3\theta_3+\theta_1\theta_3^3+\theta_2^3\theta_3$$
$$+\theta_2\theta_3^3)-5120a^2H^3(\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_1^2\theta_3+\theta_1\theta_3^2+\theta_2^2\theta_3+\theta_2\theta_3^2).$$
$$ABC=4826809a^{12}\theta_1^3\theta_2^3\theta_3^3+64000a^6H^3\theta_1^2\theta_2^2\theta_3^2-2097152H^6\theta_1\theta_2\theta_3$$
$$+270400a^8H^2\theta_1^2\theta_2^2\theta_3^2(\theta_1+\theta_2+\theta_3)+2768896a^4H^4\theta_1\theta_2\theta_3(\theta_1^2+\theta_2^2+\theta_3^2)$$
$$+1142440a^{10}H\theta_1^2\theta_2^2\theta_3^2(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)-3655808a^8H^2\theta_1\theta_2\theta_3(\theta_1^2\theta_2^2$$
$$+\theta_1^2\theta_3^2+\theta_2^2\theta_3^2)-204800a^4H^4\theta_1\theta_2\theta_3(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)$$
$$+655360a^4H^4\theta_1\theta_2\theta_3(\theta_1+\theta_2+\theta_3)-865280a^6H^3\theta_1\theta_2\theta_3(\theta_1\theta_2^2+\theta_1^2\theta_2$$
$$+\theta_1\theta_3^2+\theta_1^2\theta_3+\theta_2\theta_3^2+\theta_2^2\theta_3).$$
$$\theta_1^2+\theta_2^2+\theta_3^2=-2(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)=I/2a^2.$$
$$\theta_1^3+\theta_2^3+\theta_3^3=-3(\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_1^2\theta_3+\theta_1\theta_3^2+\theta_2^2\theta_3+\theta_2\theta_3^2)-6\theta_1\theta_2\theta_3=3\theta_1\theta_2\theta_3$$
$$=-3J/4a^3.$$
$$\theta_1^2\theta_2^2+\theta_1^2\theta_3^2+\theta_2^2\theta_3^2=I^2/16a^4.$$
$$\theta_1^3\theta_2^3+\theta_1^3\theta_3^3+\theta_2^3\theta_3^3=-I^3/64a^6+3\theta_1\theta_2\theta_3=(12J^2-I^3)/64a^6.$$
$$\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_1^2\theta_3+\theta_1\theta_3^2+\theta_2^2\theta_3+\theta_2\theta_3^2=-3\theta_1\theta_2\theta_3=3J/4a^3.$$

$$\theta_1^3\theta_2+\theta_1\theta_2^3+\theta_1^3\theta_3+\theta_1\theta_3^3+\theta_2^3\theta_3+\theta_2\theta_3^3=(I/2a^2)(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)=-I^2/8a^4.$$

$$\theta_1^3\theta_2^2+\theta_1^2\theta_2^3+\theta_1^3\theta_3^2+\theta_1^2\theta_3^3+\theta_2^3\theta_3^2+\theta_2^2\theta_3^3=-(\theta_1^2\theta_2^2\theta_3+\theta_1^2\theta_2\theta_3^2+\theta_1\theta_2^2\theta_3^2)$$
$$=-\theta_1\theta_2\theta_3(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)=-IJ/16a^5.$$

$$\therefore A+B+C=\tfrac{1}{4}(80HI-507aJ).$$

$$AB+AC+BC=\frac{1}{64a^2}(342732a^4J^2-28561a^4I^3+179456a^2H^2I^2-262144H^4I$$
$$-27040a^3HIJ-245760aH^3J).$$

$$ABC=\frac{1}{64a^3}(12943360a^3H^3J^2-4826809a^6J^3+33554432H^6J-22970368a^2H^4IJ$$
$$-1142440a^5HIJ^2+3655808a^4H^2I^3J).$$

$$\therefore \frac{a^{12}\Pi}{4096}=-[(2125764a^6J^2+524880a^5HIJ^2-3326427a^6J^2+\tfrac{1}{16}(27761292a^6J^2$$
$$-2313441a^6J^3+14535936a^4H^2I^2-21233664a^2H^4I-2190240a^5HIJ$$
$$-19906560a^3H^3J)+\tfrac{1}{16}(12943360a^3H^3J-4826809a^6J^2+33554432H^6$$
$$-22970368a^2H^4I-1142440a^5HIJ+3655808a^4H^2I^2).$$

$$\therefore a^{12}\Pi=256(2313441a^6I^3-3723875a^6J^2-5065400a^5HIJ+6963200a^3H^3J$$
$$+44204032a^2H^4I-18191744a^4H^2I^2-33554432H^6).$$

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## GEOMETRY.

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**306. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.**

Find the length of the perpendicular let fall from the point in space (5, 6, 7) upon the line $x=2z-3$, and $y=-3z+1$.

**Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Let $\left.\begin{matrix}x=sz+p\\y=tz+q\end{matrix}\right\}$ be the line in space, and $m$, $n$, $l$ the point in space. $m$, $n$, $l=A$, the vertex of a triangle; $p$, $q$, $0=B$, the point where line pierces the $xy$ plane; $\frac{pt-qs}{t}$, 0, $-\frac{q}{t}=C$, the point where line pierces the $xz$ plane.

$$c=AB=\sqrt{[(m-p)^2+(n-q)^2+l^2]}.$$

$$b=AC=\sqrt{\left(\frac{mt-pt+qs}{t}\right)^2+n^2+\left(\frac{lt+q}{t}\right)^2}.$$

$$a=BC=(q/t)\sqrt{(1+s^2+t^2)}.$$

The perpendicular from the vertex $A$ on the side $BC$ is

$$p=\sqrt{c^2-\left(\frac{a^2+c^2-b^2}{2a}\right)^2}.$$

$$\therefore p=\sqrt{(m-p)^2+(n-q)^2+l^2-\left(\frac{(m-p)s+(n-q)t+l}{\sqrt{(1+s^2+t^2)}}\right)^2}$$

$$= \sqrt{\frac{(mt-pt-ns+qs)^2+(sl-m+p)^2+(tl-n+q)^2}{1+s^2+t^2}}.$$

In the problem, $m=5$, $n=6$, $l=7$, $s=2$, $t=-3$, $p=-3$, $q=1$.

$$\therefore\ p = \sqrt{\frac{(34)^2+(6)^2+(26)^2}{14}} = \tfrac{1}{14}\sqrt{(6538)} = 11.55.$$

Also solved by Mary R. Beck, A. H. Holmes, and J. Scheffer, and the Proposer.

307. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

A family of planes containing a common line intersects a sphere. Find the orthogonal trajectories of the traces. An analytic solution is preferred.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let the sphere and family of planes be respectively,

$$x^2+y^2+z^2-2az=0,\ z+c=\lambda x \ldots (1),$$

in which $\lambda$ is the parameter of the planes, so that the common line is

$$z+c=x=0 \ldots (2).$$

The following equations define an inversion in space,

$$x=\frac{2a^2x'}{x'^2+y'^2+z'^2},\quad y=\frac{2a^2y'}{x'^2+y'^2+z'^2},\quad z=\frac{2a^2z'}{x'^2+y'^2+z'^2} \ldots (3).$$

The result of (3) applied to (1) is, after reduction,

$$z'-a=0,\quad c(x'^2+y'^2)+ca^2+2a^3-2a^2\lambda x'=0 \ldots (4).$$

Thus the circles defined by (1) have become the circles in (4), all lying in one plane. The family of circles orthogonal to (4) is

$$z'-a=0,\quad c(x'^2+y'^2)-ca^2-2a^3-2a^2\mu y'=0 \ldots (5),$$

in which $\mu$ is the parameter.

Solving (3) for $x'$, $y'$, $z'$, gives formulas for inverting (4) and (5). The former, of course, becomes (1) again, while the latter becomes

$$x^2+y^2+z^2-2az=0,\quad z(a+c)-ac+\mu ay=0 \ldots (6).$$

It is a property of inversion that angles remain invariant, so that the system (1) and (6) is orthogonal and (6) is the desired trajectory, composed of circles of which the respective planes have the common line

$$z(a+c)-ac=y=0 \ldots (7).$$

The inversion of (3) was so chosen that the original sphere became a plane, thus making the solution depend upon the simpler problem of finding the orthogonal trajectory of a family of plane curves.

Also solved by G. B. M. Zerr.

## CALCULUS.

**81. Proposed by J. OWEN MAHONEY, M. Sc., Dallas, Texas.**

Solve, $y^2\frac{d^2y}{dx^2}+a\frac{dy}{dx}=bx$.

**Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Let $y=A+Bx+Cx^2+Dx^3+Ex^4+\ldots$

$dy/dx=B+2Cx+3Dx^2+4Ex^3+55Fx^4+\ldots$

$d^2y/dx^2=2C+6Dx+12Ex^2+20Fx^3+30Gx^4+\ldots$

$y^2=A^2+B^2x^2+C^2x^4+2ABx+2ACx^2+2ADx^3+2AEx^4+2BCx^3+2BDx^4+\ldots$

$\therefore y^2(d^2y/dx^2)+a(dy/dy)^2=bx$ gives us

| | | | | | |
|---|---|---|---|---|---|
| $bx=2A^2C$ | $+6A^2D$ | $x+12A^2E$ | $x^2+20A^2F$ | $x^3+30A^2G$ | $x^4+\ldots$ |
| $+aB^2$ | $+4ABC$ | $+2B^2C$ | $+6B^2D$ | $+12B^2E$ | |
| | $+4aBC$ | $+12ABD$ | $+24ABE$ | $+2C^3$ | |
| | | $+4AC^2$ | $+16ACD$ | $+40ABF$ | |
| | | $+4aC^2$ | $+4BC^2$ | $+28ACE$ | |
| | | $+6aBD$ | $+8aBE$ | $+12AD^2$ | |
| | | | $+12aCD$ | $+16BCD$ | |
| | | | | $+9aD^2$ | |
| | | | | $+10aBF$ | |
| | | | | $+16aCE$ | |

Equating like powers of $x$ we get

$$C=-\frac{aB^2}{2A^2},\quad D=\frac{2A^2b+4aAB^3+4a^2B^3}{12A^4},$$

$$E=\frac{aB^4[A^2-(A+a)(4A+3a)]-abA^2B-2bA^3B}{12A^6}.$$

$$\therefore y=A+Bx-\frac{aB^2}{2A^2}x^2+\frac{2A^2b+4AB^3+4a^2B^3}{12A^4}x^3$$

$$+\frac{aB^4[A^2-(A+a)(4A+3a)-abA^2B-2bA^3B}{12A^6}x^4+\ldots$$

where $A$ and $B$ are constants of integration.

This solution does not give a unique result.

233. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}dx$, and (b) $\int_0^{\frac{1}{2}\pi} \frac{\sin^2 nx}{\sin x}dx$, where $n$ is a positive integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. SCHEFFER, A. M., Hagerstown, Md.

$$(a)\ \frac{\sin nx}{\sin x} = 2[\cos x + \cos 3x + \cos 5x + \ldots + \cos(n-1)x],\ n \text{ even},$$

$$= 1 + 2[\cos 2x + \cos 4x + \cos 6x + \ldots + \cos(n-1)x],\ n \text{ odd}.$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}dx = 2\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots + \frac{1}{n-1}\sin(n-1)x\right)_0^{\frac{1}{2}\pi}$$

$$= 2\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ldots \pm \frac{1}{n-1}\right),\ n \text{ even} = \tfrac{1}{2}\pi \text{ if } n = \infty.$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}dx = \left(x + \sin 2x + \tfrac{1}{2}\sin 4x + \tfrac{1}{3}\sin 6x + \ldots \frac{2}{n-1}\sin(n-1)x\right)_0^{\frac{1}{2}\pi}$$

$$= \tfrac{1}{2}\pi \text{ when } n \text{ is odd}.$$

$$(b)\ \frac{\sin^2 nx}{\sin x} = \sin x + \sin 3x + \sin 5x + \ldots + \sin(2n-1)x.$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 nx}{\sin x}dx = -\left(\cos x + \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x + \ldots + \frac{1}{2n-1}\cos(2n-1)x\right)_0^{\frac{1}{2}\pi}$$

$$= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \ldots + \frac{1}{2n-1}.$$

Also solved by Prof. F. Anderegg.

## DIOPHANTINE ANALYSIS.

139. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

$2^{n-1}(2^n-1)$ is a multiply perfect number of multiplicity 2 when $2^n-1$ is prime. Prove that there are no other multiply perfect numbers containing only 2 distinct primes.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $2^n-1=b$.

Then $\frac{2^n-1}{2^{n-1}(2-1)} \cdot \frac{b^2-1}{b(b-1)} = \frac{2^n-1}{2^{n-1}} \cdot \frac{b+1}{b} = \frac{2^n-1}{2^{n-1}} \cdot \frac{2^n}{2^n-1} = 2.$

$\therefore 2^{n-1}(2^n-1)$ is a multiply perfect number of multiplicity 2. The second part of the problem is demonstrated in problem 137, Vol. XIII, Nos. 8-9.

140. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Determine (any way) whether the Diophantine equation $\left(\frac{2x-1}{3}\right)=x^2+y^2$ has any positive integer solutions.

Solution by JACOB WESTLUND, Ph. D., Purdue University.

In order that $\frac{2x-1}{3}$ shall be an integer we must have $x=2+3a$, where $a$ is a positive integer. Hence $(1+2a)^3=(2+3a)^2+y^2$ or after a few reductions $y^2=8a^3-6a+3(a^2-1)$.

If $a$ is odd, this equation is impossible, since in that case $y$ must be even and hence all the terms except $6a$ divisible by 4.

If $a$ is even, we put the equation in the form $y^2=8a^3+3a(a-2)-3$. This shows that $y$ must be odd and $y^2+3$ divisible by 8. Hence, setting $y=2b+1$, $4b^2+4b+4$ should be divisible by 8 or $b(b+1)+1$ divisible by 2, which is impossible. Hence the given equation has no positive integer solutions.

Also solved by A. H. Holmes.

No solution of 141 has yet been received.

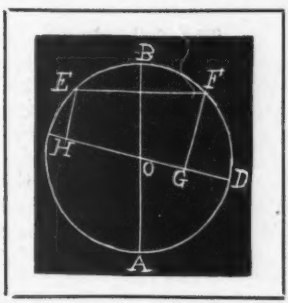
## AVERAGE AND PROBABILITY.

178. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random planes cut a given sphere. What is the chance that they intersect within the sphere?

I. Solution by HENRY HEATON, Belfield, N. D.

Let $AB$ and $CD$ be axes of the sphere perpendicular to the two planes and let $EF$ be a trace of one of the planes.




Put $x=OI$, the distance of the plane through $EF$ from the center of the sphere. Put $\theta=\angle BOC$. Then $HG$, the projection of $EF$ upon $CD=2\sqrt{(a^2-x^2)}\sin\theta$.

It seems to be generally understood that the number of directions of the plane perpendicular to $CD$ depends upon the number of different directions possible to $CD$, and that this depends upon the number of points in the surface of the sphere. Hence the number of planes of the direction $\theta$ is proportional to $\sin\theta$. The angle $\theta$ being supposed fixed the chance of intersection within the sphere is $\frac{HG}{CD}=\frac{\sqrt{(a^2-x^2)}\sin\theta}{a}$.

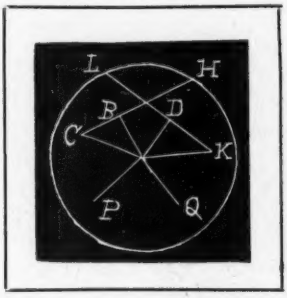
Hence the required probability is

$$p=\int_0^{\frac{1}{2}\pi}\int_0^a \sqrt{(a^2-x^2)}\sin^2\theta\,d\theta\,dx \Big/ \int_0^{\frac{1}{2}\pi}\int_0^a a\sin\theta\,d\theta\,dx$$

$$=\frac{\pi}{4}\int_0^{\frac{1}{2}\pi}\sin^2\theta\,d\theta\ /\int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta=\frac{\pi^2}{16}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $CH$, $LK$ be the diameters of the sections of the sphere made by the planes. $B$, $D$ their centers; $O$ the center of the sphere; $OQ$ a line such that a line in the plane $LK$ is parallel to the plane $DOQ$. $OC=OK=r$, $\angle COB=\theta$, $\angle KOD=\phi$, $\angle DOQ=\psi$. The limits of $\theta$ and 0 are $\frac{1}{2}\pi$; of $\phi$, 0 and $\frac{1}{2}\pi$; of $\psi$ $\pm(\theta-\phi)$ and $\theta+\phi$. The double sign is used $+$ for $\theta>\phi$, $-$ for $\theta<\phi$. Hence the chance $p$ is




$$p=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi}d\theta\,d\phi\,d\psi\Big/\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}d\theta\,d\phi\,d\psi$$

$$=\frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi}d\theta\,d\phi\,d\psi=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\phi\,d\theta\,d\phi=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}d\theta=\frac{1}{2}.$$

NOTE.—These two solutions differ in the method of distributing the direction of the random planes. ED. F.

## NOTES AND NEWS.

Professor C. Alasia, mathematical editor of the "Rivista di Fisica e Matematica," of Pisa and Pavia, will review in that journal all new publications sent to him at Ozieri, Italy.

At the University of Chicago, Assistant Professor L. E. Dickson has been promoted to an associate professorship in mathematics, and Associate Professor Heinrich Maschke to a full professorship in mathematics.

The following courses in Mathematics and Mathematical Astronomy are to be given at the University of Chicago during the Summer Quarter of 1907 beginning June 15th: By Professor Moore: Graphical Methods in Algebra especially for teachers, 4 hours; Theory of Determinants, Advanced Course, 4 hours; General Seminar, 2 hours. By Professor Bolza: Theory of Functions of Complex Variables, 4 hours; Problems in Theory of Functions, 2 hours; Abelian Functions, 2 hours. By Assistant Professor Slaught: Integral Calculus, 5 hours; Differential Equations, 4 hours. By Associate Professor Dickson: Trigonometry, 5 hours; Solid Analytical Geometry, 5 hours; Continuous Groups, 4 hours. By Assistant Professor Moulton: Descriptive Astronomy, 5 hours; Introduction to Celestial Mechanics, 4 hours. By Assistant Professor Laves: Descriptive Astronomy, 5 hours; General Astronomy and Observatory Work, 5 hours. By Dr. Lunn: Curve Tracing and Differential Calculus, 5 hours; Dynamics of Oscillatory Systems, 4 hours. By Mr. Lennes: Plane Analytic Geometry, 5 hours; Critical Review of Secondary Mathematics, 4 hours. In the College of Education: By Professor Myers: Pedagogy of Elementary School Mathematics, Pedagogy of Secondary School Mathematics.

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# THE AMERICAN MATHEMATICAL MONTHLY.

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## CO-POLAR AND CO-AXIAL TRIANGLES IN CONICS.


By REV. ALAN S. HAWKESWORTH, Pittsburg, Penn.


*Theorem 1.* If an inscribed and a circumscribed triangle to any conic curve touch in the same three points, then will they be co-polar in a point within the curve; and co-axial in the polar of said point [Figs. 1 and 2].

Draw the tangents $PR'Q'$, $R'QP'$, and $RP'Q'$, of any three points $PQR$ upon any conic curve, meeting each other, and forming the circumscribing triangle $P'Q'R'$. And join $PQR$, forming an inscribed triangle, which touches the conic in the same three points as $P'Q'R'$. Draw $Cp$, $Cq$, and $Cr$, the semi-diameters in ellipse [Fig. 1], hyperbola, or circle, parallel to the tangents at $P$, $Q$, and $R$, respectively. Or in the parabola, the lines from any point $C$ to the axis, or any diameter, parallel to said tangents.

Then, by a well known theorem, $PR'=R'Q=Cp : Cq$, and $RQ'=PQ'=Cr : Cp$, and $QP'=P'R=Cq : Cr$. So that, multiplying and cancelling, $PR'.RQ'.QP'=R'Q.PQ'.P'R$. And therefore, by a known theorem in "Transversals," lines $PP'$, $QQ'$, and $RR'$ must concur in a point $O$, or the inscribed and circumscribed triangles $PQR$, and $P'Q'R'$ be co-polar in $O$.

But as so co-polar, they must also be co-axial, by a known theorem, a fact also evident from "Poles and Polars." For if chords $PQ$ and $RQ$ cut the tangent of $R$ and $P$ in $J$ and $I$, respectivly, while line $JI$ and chord $PR$ concur in $H$, and $HQ$ be joined, then will the three summits of quadrangle $PIJR$ be $H$, $Q'$, and $Q$ on the curve. So that $Q'Q$ is the polar of $H$; and $HQ$ thus the tangent at $Q$, identical with $R'QP'$.

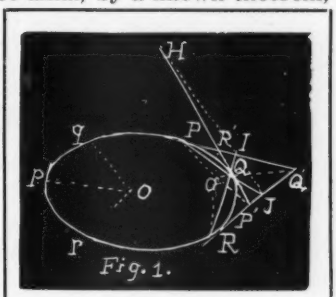

Fig. 1.

Next, $HIJ$, the co-axial line of triangles $PQR$ and $P'Q'R'$, is also the polar, in respect to the conic, of $O$, their co-polar point. For $RP'JQ'$ being a harmonic range of quadrangle $PIJR$; $PR$, $POP'$, $PQJ$ and $PIQ'$ is also a harmonic pencil; and $POP'$ thus the polar of $I$, and its chord of tangential contact. Similarly, $PR'IQ'$ being a harmonic range, gives us